110 Review Notes

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Chapter 3: Linear Maps

- - $S \in \mathcal{L}(V), S = AB$ invertible $\iff$ A invertible and B invertible.
  - In the general case: $S = AB$ is invertible $\iff$ A onto and B one-to-one.
- For $S \in \mathcal{L}(U, V)$: $\exists T$ such that $Tu = Su$, $\forall u \in U$ $\iff$ S one-to-one.
- Homework five 3.D.4 3.D.5.
  - There exists invertible S such that $T_1 = ST_2$ $\iff$ $\operatorname{null} T_1 = \operatorname{null} T_2$.
  - There exists invertible S such that $T_1 = T_2 S$ $\iff$ $\operatorname{range} T_1 = \operatorname{range} T_2$.
- $T_1 = T_2$ $\iff$ $T_1 v_i = T_2 v_i$ for any basis vector v_i.
- $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are T-invariant.
- The preimages of linear independent vectors in $\operatorname{range} T$ are linearly independent.
- $v \in U$ $\iff$ $v + U = 0$ in V/U; $v - w \in U$ $\iff$ $v + W = u + W$.
- $v + U = x + W$ $\implies$ $U = W$
- $\dim \operatorname{range} A = \dim \operatorname{range} A^* = \dim \operatorname{range} A^T$
- The quotient map (canonical map) is surjective, and its null space is U.
  - $\pi : W \to V/U$, $w \mapsto w + U$. $\operatorname{null} \pi = W \cap U$

Chapter 5: Eigen-stuffs

- T invertible $\iff$ 0 is not an eigenvalue of $T \iff \det T \neq 0$.
- The following statements regarding diagonalization are equivalent:
  - T is diagonalizable.
  - V has an eigenbasis with respect to T.
  - $V = \bigoplus_{\lambda \in \mathbf{F}} E(\lambda, T)$.
  - $\dim V = \sum_{\lambda \in \mathbf{F}} \dim E(\lambda, T)$.
  - There exists 1-dimensional invariant subspaces under T such that

    $$V = \bigoplus_{i=1}^{\dim V} U_i.$$

  - $\forall \lambda$, $G(\lambda, T) = E(\lambda, T)$, aka every generalized eigenvector of T is an eigenvector of T.
  - (Complex vector space only) the minimal polynomial of T has no repeated zeros.
- α is an eigenvalue of $p(T) \iff \alpha = p(\lambda)$, where λ is an eigenvalue of T. Therefore if λ has corresponding eigenvector v, then $p(T)v = p(\lambda)v$.

Chapter 6: Inner Product Spaces

- Additivity is still preserved in the second slot. However, $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$.
- $\langle u, v\rangle = 0 \iff \|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$.
- $v = 0 \iff \langle v, w\rangle = 0$ for all w. Similarly, $u = v \iff \langle u, w\rangle = \langle v, w\rangle$ for all w.
- (5.B.4) $P^2 = P \iff V = \text{null } P \oplus \text{range}\, P \;\wedge\; P|_{\text{range}\, P} = I \;\wedge\; P|_{\text{null}\, P} = 0$.
- Know $P^2 = P$, prove there exists a subspace U such that $P = P_U \iff$ To prove a projection is orthogonal $\iff$ Prove null$P$ $\perp$ range$P$, which implies $(\text{null } P)^{\perp}$ = range P and vice versa because the dimension add up. **In other words**, $P^2 = P$, $P = P_U \iff \text{range}\, P \perp \text{null}\, P$

Chapter 7: Operators on Inner Product Spaces

Self-adjoint and Normal Operators

- Definition: $\langle Tv, w\rangle = \langle v, T^*w\rangle$. Self adjoint: $\langle Tv, w\rangle = \langle v, Tw\rangle$. Normal: $TT^* = T^*T$ (commutes with its adjoint).
- T invertible $\iff$ T^* invertible. *Prove using properties of null and range of T^*, i.e. 7.7.*
- $M(T) = \overline{M(T^*)}$ only when orthonormal basis.
- Every operator is the sum of a self-adjoint operator and a normal operator.

$$T = \frac{T + T^*}{2} + \frac{T - T^*}{2}$$

- The eigenvectors that correspond with distinct eigenvalues of T are linearly independent; if T is ***normal***, then the eigenvectors of T that correspond to **distinct** eigenvalues are not only linearly independent, but also ***orthogonal***.
- (7.A.3) $T \in L(V)$, U is invariant under $T \iff U^\perp$ is invariant under T^*.
- (Two more in 9.30) T is **normal**, and U is invariant under T, then
  - U is invariant under T^*.
  - $U^\perp$ is invariant under T.
- (7.21, 7.A.16, 7.A.17) T normal, then:
  - T and T^* have the same eigenvectors with conjugate eigenvalues ($\lambda \longleftrightarrow \bar{\lambda}$).
  - $\operatorname{range} T = \operatorname{range} T^*$.
  - $\operatorname{range} T = \operatorname{range} T^k$, $\operatorname{null} T = \operatorname{null} T^k$.

The Spectral Theorem

In the complex space:

- T normal $\iff$ V has orthonormal eigenbasis with respect to T.
- T self-adjoint $\iff$ V has orthonormal eigenbasis with respect to **normal** T with *all real* eigenvalues.

In the real space:

- T self-adjoint $\iff$ T has orthonormal eigenbasis (with all real eigenvalues, because we're in the real space).

Positive Operators and Isometries

- (7.35) The following are equivalent:
  - T is positive.
  - T is self-adjoint and $\langle Tv, v\rangle \geq 0$ for all $v \in V$.
  - T is self-adjoint and all eigenvalues are non-negative reals.
  - There exists an operator R such that $T = R^*R$ or RR^*.
  - T has a positive or self-adjoint square root.
- (7.C.7) The following are equivalent:
  - T is positive and invertible.
  - T is self-adjoint and $\langle Tv, v\rangle > 0$ for all $v \neq 0$.
  - T is self-adjoint and all eigenvalues of T are strictly greater than zero.
- The following are equivalent:
  - S is an isometry.
  - $\|v\| = \|Sv\|$.
  - S takes one or all orthonormal basis/bases to an orthonormal basis.

Chapter 8: Generalized Eigen-stuffs

$\mathbf{F} = \mathbf{C}$ throughout this chapter.

- Key Idea:
$$V = \bigoplus_\lambda G(\lambda, T);\ T = \bigoplus_\lambda T|_{G(\lambda,T)}.$$
- If N is nilpotent, then
  - $N^{\dim V} = 0$.
  - 0 is the only eigenvalue of N.

- N has a strictly upper triangular matrix with respect to some bases.

- For every basis of V for which T has an upper-triangular matrix, the number of times an eigenvalue λ appears on the diagonal $= \dim G(\lambda, T)$, aka the multiplicity of λ.

- $G(\lambda, T) = G(\lambda^{-1}, T^{-1})$

- The null space keeps on growing until it stops in at most $\dim V$, and then it stops once and for all. Therefore to prove $\operatorname{null} T = \operatorname{null} T^{\dim V} \iff \operatorname{null} T^2 \subseteq \operatorname{null} T$.

Characteristic and Minimal Polynomials

- If $p(T) = 0$, then any eigenvector of T is a root of p.

- $q(T) = 0 \iff q$ is a multiple of the minimal polynomial of T.

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  - $p(T)$ is nilpotent and $p(x)$ has no real zeros $\implies T$ has no real eigenvalues.

    Proof. $(p(T))^n$ is a multiple of the minimal polynomial of T, which has no real zeros. □

  - The discriminant $\Delta \geq 0$ of a degree-two p and $p(T) = 0 \iff T$ has a real eigenvalue.

- $\dim \operatorname{span}(I, A, A^2, \cdots) = \deg p$, where p is the minimal polynomial of A.
  `https://math.stackexchange.com/questions/79283`

Chapter 9: Operators on Real Vectors Spaces

$\mathbf{F} = \mathbf{R}$ throughout this chapter.

Complexification

- The eigenvectors of T are the real eigenvectors of $T_{\mathbf{C}}$.

- For an operator T over a real vector space:

  - If T is on an odd-dimensional real vector space, then T has an eigenvalue.
  - If T does not have an eigenvalue, then T is on an even-dimensional real vector space.

- - If T is normal, then $T_{\mathbf{C}}$ is normal.
    `https://piazza.com/class/k31w9e2vrbk43x?cid=567`
  - If T is self-adjoint, then $T_{\mathbf{C}}$ is self-adjoint.
    `https://math.stackexchange.com/questions/1887417`

Misc

- State the things that you know and the statement you want to prove. Construct a bridge between them from both sides.
- Go to transformations for counterexamples/nilpotent operators: Rotations, zero transformations.
- Have a basis on V, define inner product as the *dot product*, thus V magically becomes an inner product space and the basis is magically orthonormal.
- Prove $v = w \iff v - w = 0$.
- Prove p if and only if $q \iff$ Prove $p \Rightarrow q$ and $!p \Rightarrow !q$
- $p \mid q \iff$ Let $q = pt + r$, $r = 0$.
- Want to prove $\dim \operatorname{null} T =$? Use rank-nullity theorem. Have something to do with $\dim \operatorname{range} T$? Use rank-nullity theorem. When in doubt, use rank-nullity theorem.
- Turn inner product spaces related questions into matrix questions by choosing an orthonormal basis.
- Want something to satisfy every vector in the entire vector space? Just make sure it satisfy all the basis vectors.
- $V = W \iff V \subseteq W \wedge \dim V = \dim W$.
- $\operatorname{range}(ST) \subseteq \operatorname{range} S$.